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The formal specification of the elements of stress in cartesian, and in cylindrical and spherical polar coordinates.

BY

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X. The formal specification of the elements of stress in cartesian, and in cylindrical and spherical polar coordinates.

By R. F. GWYTHER, M.A.

(*Received March 8th, 1912. Read March 19th, 1912.*)

This paper is a prefatory paper, as it is concerned with the collection of material necessary for the application of a novel method to the investigation of stresses in material structures.

The interest no doubt begins with the applications, but no such application accompanies this part of an uncompleted investigation.

The usual method in Elasticity has been to assume from the beginning that Hooke's Law is obeyed, and thus to make the equations of displacement replace the equations of stress, and so become the fundamental subject of investigation, the question of surface-tractions being considered later.

The method I now propose follows a different order of investigation, the equations of stress being solved as the first step. Such solutions are then to be conditioned so as to be able to satisfy the surface-tractions. Lastly, the question whether the solution can be made to satisfy Hooke's Law is to be taken as the final condition.

It is possible that the application of this final condition may indicate to what extent and in what way the failure to follow Hooke's Law may show itself.

May 13th. 1912.

Cartesian Coordinates.

The equations of stress, with the usual notation, are

$$\frac{dP}{dx}+\frac{dU}{dy}+\frac{dT}{dz}=X,$$

$$\frac{dU}{dx}+\frac{dQ}{dy}+\frac{dS}{dz}=Y,$$

$$\frac{dT}{dx}+\frac{dS}{dy}+\frac{dR}{dz}=Z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where X, Y, Z are to include 'inertia terms' as well as forces.

I shall assume that we may write

$$X=\frac{d\phi}{dx}+\frac{dN}{dy}-\frac{dM}{dz},$$

$$Y=\frac{d\phi}{dy}+\frac{dL}{dz}-\frac{dN}{dx},$$

$$Z=\frac{d\phi}{dz}+\frac{dM}{dx}-\frac{dL}{dy} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

The equations (1) can, of course, only have solutions which are to some extent indefinite, but the *form* of the solutions may be quite definite, and it is proposed to find these formal solutions or specifications of stress.

We shall, in the first place, find *particular integrals*, and then the *complementary functions*, to use the terminology of ordinary differential equations.

When the values of X, Y and Z are given in any special case, the particular integrals will probably be readily found. But for the purpose of what follows later, it will be well to give a general method of procedure, although that method need not be followed in each special case.

The particular integral solution.

Assume

$$P_0=\phi_1,\quad S_0=0,$$

$$Q_0=\phi_2,\quad T_0=0,$$

$$R_0=\phi_3,\quad U_0=0;$$

and

$$P_1 = 2\frac{dv}{dy} + 2\frac{dw}{dz},$$

$$Q_1 = 2\frac{du}{dx} + 2\frac{dw}{dz},$$

$$R_1 = 2\frac{du}{dx} + 2\frac{dv}{dy},$$

$$S_1 = -\frac{dw}{dy} - \frac{dv}{dz},$$

$$T_1 = -\frac{du}{dz} - \frac{dw}{dx},$$

$$U_1 = -\frac{dv}{dx} - \frac{du}{dy}, \quad \ldots\ldots\ldots \quad (3).$$

Then we shall require that

$$\frac{d\phi_1}{dx}=\frac{d\phi}{dx},\quad \frac{d\phi_2}{dy}=\frac{d\phi}{dy},\quad \frac{d\phi_3}{dz}=\frac{d\phi}{dz};$$

and

$$\frac{dw}{dy}-\frac{dv}{dz}=L,\quad \frac{du}{dz}-\frac{dw}{dx}=M,\quad \frac{dv}{dx}-\frac{du}{dy}=N, \qquad (4),$$

as the conditions for satisfying (1) and (2).

I have retained ϕ_1, ϕ_2, ϕ_3, with their definition in (4), because, although we might replace each of them by ϕ. The single function ϕ is ordinarily introduced, partly for the sake of form and partly because in ordinary dynamical problems it represents a form of "energy," but in the case under consideration these arguments are void, and each of the quantities ϕ_1, ϕ_2, ϕ_3 will have its simplest value. For an illustration, note a heavy body rotating about a vertical axis, in which case the different values of ϕ_1, ϕ_2 and ϕ_3 are too obvious to require description.

If the single function should be retained, it would be necessary to *correct* by terms from the complementary function to which we now proceed.

The Complementary Function Solution.

The right hand side of equations (1) must for this purpose be replaced by zero, and the solution should contain *six* arbitrary functions.

If we replace each of the stresses by the general linear expression of second-differential coefficients of a function, we shall have 6 arbitrary constants in the expression for each stress, and therefore 36 such constants altogether.

Substituting in the stress equations, we should obtain three linear expressions of third-differential coefficients, each of which is to vanish, and we should therefore obtain 30 linear equations of condition between the arbitrary constants, leaving *six* independent and arbitrary constants.

As in the ordinary theory of differential equations, each independent arbitrary constant corresponds to an arbitrary function. This plan therefore leads to the complete solution or specification required.

The labour is made quite slight by the consideration of the stress equations to be solved: for we may conclude; —(*a*) that P will contain no differential coefficient in x, Q none in y, and R none in z;—(*b*) that S will contain no second-differential coefficient in either y or z, T none in either z or x, U none in either x or y.

The solution is

$$
\begin{aligned}
P_2 &= \frac{d^2\Theta_3}{dy^2} - 2\frac{d^2\Psi_1}{dydz} + \frac{d^2\Theta_2}{dz^2}, \\
Q_2 &= \frac{d^2\Theta_3}{dx^2} - 2\frac{d^2\Psi_2}{dxdz} + \frac{d\,\Theta_1}{dz^2}, \\
R_2 &= \frac{d^2\Theta_2}{dx^2} - 2\frac{d^2\Psi_3}{dxdy} + \frac{d^2\Theta_1}{dy^2}, \\
S_2 &= -\frac{d^2\Psi_1}{dx^2} + \frac{d^2\Psi_2}{dxdy} + \frac{d^2\Psi_3}{dxdz} - \frac{d^2\Theta_1}{dydz}, \\
T_2 &= +\frac{d^2\Psi_1}{dxdy} - \frac{d^2\Theta_2}{dxdz} - \frac{d^2\Psi_2}{dy^2} + \frac{d^2\Psi_3}{dydz}, \\
U_2 &= -\frac{d^2\Theta_3}{dxdy} + \frac{d^2\Psi_1}{dxdz} + \frac{d^2\Psi_2}{dydz} - \frac{d^2\Psi_3}{dz^2}. \quad (5).^*
\end{aligned}
$$

The complete solution finally is

$P = P_o + P_1 + P_2$, etc.,

$S = S_1 + S_2$, etc., from (3) and (5).

We have now obtained an explicit statement for the elements of stress, applicable to all substances for which these elements are continuous functions of the coordinates. They may be taken to apply to most materials for construction, and to apply fairly well to earth pressures on a retaining wall, and less definitely to the pressures in a corn bin or bunker or in a quicksand, but for different reasons.

Avoiding such special cases, we have to consider (1) cases where Hooke's Law of connection between stress and strain holds good, (2) cases where the elastic limit is exceeded, (3) cases where a viscous flow of the material is set up, and (4) the conditions for rupture.

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[*] In a paper by Sir G. B. Airy, published as a Report by the British Association (1862, page 82), the value of a general solution is minimised. But giving the fullest value to that investigation, the solution there proposed is covered if the Θ-functions alone in (5) are retained, and the Ψ-functions ignored.

It is not proposed to consider all these cases in this paper, but only to set out general lines for investigation, taking the case of Hooke's Law as the general case, but having regard to departures from it.

If Hooke's Law held universally, we should have a second explicit expression for the stresses in terms of the strains, and ultimately in terms of the three components of the displacement.

With two explicit forms for the elements of stress, there are three modes of further investigation open :

(1) To take the Hooke's Law specification as absolute and to substitute in the statical equations.

This has been the practice.

(2) To take the statical specification as absolute and to find the cases which satisfy Hooke's Law.

(3) To combine the two specifications as may be found analytically most convenient in any case.

This last method will doubtless be found convenient in many cases, but it may be accompanied by the drawbacks which are inherent in the combination of two independent methods.

I proceed with the second method with the object of bringing into prominence the analytical connection which is now to be demonstrated between the statical expressions for the stress and the geometrical conditions which are consequent on the fulfilment of Hooke's Law.

The Geometrical conditions affecting elements of strain.

Using the adopted notation for strains, the conditions are well known to be

$$0=\frac{d^2g}{dy^2}+\frac{d^2f}{dz^2}-\frac{d^2a}{dydz},$$

with two analogous equations;

$$0=\frac{1}{2}\frac{d}{dx}\left(-\frac{da}{dx}+\frac{db}{dy}+\frac{dc}{dz}\right)-\frac{d^2e}{dydz},$$

with two analogous equations . . . . (6).

On comparison of the right-hand side of these equations with the right-hand side of the equations (5), it is seen that they become identical on the substitution of

$$\Theta_1 \text{ for } e, \quad \Theta_2 \text{ for } f, \quad \Theta_3 \text{ for } g;$$

$$2\Psi_1 \text{ for } a, \quad 2\Psi_2 \text{ for } b, \quad 2\Psi_3 \text{ for } c,$$

This general relation will be made use of at a later period in the paper.

Within the limits to which Hooke's Law applies, the elements of the strain can be replaced by a linear function of the elements of the stress with constant coefficients, and if the coefficients are allowed to be variable the substitution may be made, even though Hooke's Law no longer obtains. I shall, however, here limit myself to the case of a homogeneous isotropic elastic solid, and write

$$qe=(1+\sigma)P-\sigma(P+Q+R),$$

with two analogous relations;

$$qa=2(1+\sigma)S,$$

with two analogous relations,

where q and σ are constants.

Then for the relations affecting the stresses in such a solid, we shall have

$$0=(1+\sigma)\left\{\frac{d^2R}{dy^2}+\frac{d^2Q}{dz^2}-2\frac{d^2S}{dydz}\right\}-\sigma\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\left\{P+Q+R\right\},$$

with two analogous relations,

$$0=(1+\sigma)\left\{\frac{d}{dx}\left(-\frac{dS}{dx}+\frac{dT}{dy}+\frac{dU}{dz}\right)-\frac{d^2P}{dydz}\right\}+\sigma\frac{d^2}{dydz}\left\{P+Q+R\right\},$$

with two analogous relations . . . . (7).

Taking into consideration the three forms of solution (from (3) and (5)), we obtain as the final relations for a homogeneous isotropic elastic solid,

$$(1+\sigma)\left\{\frac{d^2R_2}{dy^2}+\frac{d^2Q_2}{dz^2}-2\frac{d^2S_2}{dydz}\right\}-\sigma\left(\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\left\{P_2+Q_2+R_2\right\}$$

$$+\frac{1-\sigma}{2}\left(\frac{d}{dy^2}+\frac{d^2}{dz^2}\right)\left\{P_1+Q_1+R_1\right\}$$

$$-\frac{d^2}{dy^2}\left\{\sigma(P_o+Q_o)-R_o\right\}-\frac{d^2}{dz^2}\left\{\sigma(P_o+R_o)-Q_o\right\}=0$$

with two analogous equations ;

and

$$(1+\sigma)\left\{\frac{d}{dx}\left(-\frac{dS_2}{dx}+\frac{dT_2}{dy}+\frac{dU_2}{dz}\right)-\frac{d^2P_2}{dydz}\right\}+\sigma\frac{d^2}{dydz}\left\{P_2+Q_2+R_2\right\}$$

$$-\frac{1-\sigma}{2}\frac{d^2}{dydz}\left\{P_1+Q_1+R_1\right\}$$

$$+\frac{d}{dydz}\left\{\sigma(Q_o+R_o)-P_o\right\}=0$$

with two analogous equations . . . (8).

In order to make use of these solutions, the surface-traction conditions must be employed, and unless these are readily applicable and tend to simplify the arbitrary functions, it would appear that a different choice of coordinates should here be made. Generally, the cartesian system of coordinates will only suit slab-sided figures.

I now proceed to consider other coordinate systems, but I only propose in each case to deal with the *complementary function* solution, as the *particular integral* solution can be treated in each particular case as I have already shown for cartesian coordinates.

Cylindrical polar coordinates.

The stress equations in this case are

$$\frac{1}{r}\frac{d}{dr}(Pr)+\frac{1}{r}\frac{dU}{d\theta}+\frac{dT}{dz}-\frac{Q}{r}=X,$$

$$\frac{1}{r^2}\frac{d}{dr}(Ur^2)+\frac{1}{r}\frac{dQ}{d\theta}+\frac{dS}{dz}=Y,$$

$$\frac{1}{r}\frac{d}{dr}(Tr)+\frac{1}{r}\frac{dS}{d\theta}+\frac{dR}{dz}=Z \quad . \quad . \quad . \quad . \quad (9).$$

With the guidance given by having obtained the *complementary function* solution in cartesians, it is not difficult to determine the corresponding solution in this case, namely—

$$P_2=\frac{1}{r^2}\frac{d^2\Theta_3}{d\theta^2}-\frac{2}{r}\frac{d^2\Psi_1}{d\theta dz}+\frac{d^2\Theta_2}{dz^2}+\frac{1}{r}\frac{d\Theta_3}{dr}-\frac{2}{r}\frac{d\Psi_2}{dz},$$

$$Q_2=\frac{d^2\Theta_3}{dr^2}-2\frac{d^2\Psi_2}{drdz}+\frac{d^2\Theta_1}{dz^2},$$

$$R_2=\frac{d^2\Theta_2}{dr^2}-\frac{2}{r}\frac{d^2\Psi_3}{drd\theta}+\frac{1}{r^2}\frac{d^2\Theta_1}{d\theta^2}+\frac{2}{r}\frac{d\Theta_2}{dr}-\frac{1}{r}\frac{d\Theta_1}{dr}-\frac{2}{r^2}\frac{d\Psi_3}{d\theta},$$

$$S_2=-\frac{d^2\Psi_1}{dr^2}+\frac{1}{r}\frac{d^2\Psi_2}{drd\theta}+\frac{d^2\Psi_3}{drdz}-\frac{1}{r}\frac{d^2\Theta_1}{d\theta dz}-\frac{1}{r}\frac{d\Psi_1}{dr}-\frac{1}{r^2}\frac{d\Psi_2}{d\theta}+\frac{2}{r}\frac{d\Psi_3}{dz}+\frac{\Psi_1}{r^2},$$

$$T_2=\frac{1}{r}\frac{d^2\Psi_1}{drd\theta}-\frac{1}{r^2}\frac{d^2\Psi_2}{d\theta^2}+\frac{1}{r}\frac{d^2\Psi_3}{d\theta dz}-\frac{d^2\Theta_2}{drdz}+\frac{1}{r^2}\frac{d\Psi_1}{d\theta}+\frac{1}{r}\frac{d\Theta_1}{dz}-\frac{1}{r}\frac{d\Theta_2}{dz},$$

$$U_2=\frac{d^2\Psi_1}{drdz}+\frac{1}{r}\frac{d^2\Psi_2}{d\theta dz}-\frac{d^2\Psi_3}{dz^2}-\frac{1}{r}\frac{d^2\Theta_3}{drd\theta}+\frac{1}{r^2}\frac{d\Theta_3}{d\theta}-\frac{1}{r}\frac{d\Psi_1}{dz} \quad . \quad . \quad . \quad (10).$$

In accordance with the relation noticed in cartesians between (5) and (6), we may similarly deduce from (10) the relations connecting the elements of the strain.

It seems unnecessary to reproduce either this system of equations, or the system of equations corresponding with (7) or (8).

If the complete circumstances involve symmetry about the cylindrical axis, the whole system is simplified by the independence of all functions of the coordinate θ.

Spherical Polar Coordinates.

In a paper* read before the Society a year ago, I proposed an alteration in the representation of the displacements for such a system of coordinates, the proposed form being

$$u, \quad \sin\theta\, v, \quad \sin\theta\, w.$$

The elements of the strain consequently may be written

$$\{e, \ f, \ g, \quad a, \quad b\sin\theta, \quad c\sin\theta\}.$$

and those of the stress

$$\{P, \ Q, \ R, \quad S, \quad T\sin\theta, \quad U\sin\theta\}.$$

I shall assume that this system is employed, and shall write, as is usual, x in place of $\cos\theta$.

The equations of stress then become

$$\frac{1}{r^2}\frac{d}{dr}(Pr^2) - \frac{1}{r}\frac{d}{dx}\Big\{(1-x^2)U\Big\} + \frac{1}{r}\frac{dT}{d\phi} - \frac{1}{r}(Q+R) = X,$$

$$\frac{1}{r^3}\frac{d}{dr}(Ur^3) - \frac{1}{r}\left(\frac{dQ}{dx} - \frac{Qx}{1-x^2}\right) + \frac{1}{r(1-x^2)}\frac{dS}{d\phi} - \frac{Rx}{r(1-x^2)} = \frac{Y}{\sin\theta} = Y',$$

$$\frac{1}{r^3}\frac{d}{dr}(Tr^3) - \frac{1}{r(1-x^2)}\frac{d}{dx}\Big\{(1-x^2)S\Big\} + \frac{1}{r(1-x^2)}\frac{dR}{d\phi} = \frac{Z}{\sin\theta} = Z' \quad (11).$$

The Complementary Function Solution.

The expressions corresponding with those of (5) are long, but this is inevitable in view of the generality of

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* *Manchester Memoirs*, vol. iv. (1911), No. 20.

the solution. The simplifications will not appear until a particular application is attempted, when the terms will fall into groups. The terms of the second order in the solution can be written down, and the other terms can be deduced by substitution in the stress equations. The result is as follows (12):

See Table A for equations (12).

These lead at once to the expressions connecting the strains corresponding with (6).

This paper has only dealt with the preparation of an instrument for use on simple examples, but I feel sure that it is well to have these cases treated in full. No doubt, for completion, the case of curvilinear coordinates should have been included, but this offers no considerable difficulty, and may well be deferred till the question of its use in Geostatics arises.

Although the paper contains no illustrative applications, I may perhaps be allowed to state that I have taken some steps to satisfy myself that the method proposed appears to be of practicable application to the theory of the cylindrical test-piece in Mechanical Engineering and to the theory of arches and domes in Civil Engineering.

Although I shall not consider any precise problem, each of which will require its special method of final treatment for surface-tractions, etc., I wish to show some grounds for the belief which I have expressed that the method of this paper lends itself to the treatment of particular cases.

For this purpose I have selected the case of a body symmetrical about an axis, and subject to surface-tractions only, also symmetrical about that axis. That is, I select the case of cylindrical polars, in which θ does not enter.

The expressions for the elements of stress (10) then become

$$P=\frac{d^2\Theta_2}{dz^2}+\frac{1}{r}\left(\frac{d\Theta_3}{dr}-2\frac{d\Psi_2}{dz}\right),$$

$$r\frac{dP}{dr}+P-Q=\frac{d^2}{dz^2}\left(r\frac{d\Theta_2}{dr}+\Theta_2-\Theta_1\right)=\frac{d^2\mu}{dz^2},$$

$$rR=\frac{d}{dr}\left(r\frac{d\Theta_2}{dr}+\Theta_2-\Theta_1\right)=\frac{d\mu}{dr},$$

$$rT=-\frac{d}{dz}\left(r\frac{d\Theta_2}{dr}+\Theta_2-\Theta_1\right)=-\frac{d\mu}{dz};$$

$$r^2S=-\frac{d}{dr}\left(r_2\frac{d\Psi_1}{dr}-r\Psi_1-r^2\frac{d\Psi_3}{dz}\right)=-\frac{d\lambda}{dr},$$

$$r^2U=\frac{d}{dz}\left(r^2\frac{d\Psi_1}{dr}-r\Psi_1-r^2\frac{d\Psi_3}{dz}\right)=\frac{d\lambda}{dz}\quad . \quad . \quad . \quad (13).$$

These conditions are required by Statics, and they cover the whole of the statical requirements.

If the stresses and strains follow Hooke's Law, we obtain the following independent conditions:

$$(1+\sigma)\left\{\frac{d^2Q}{dz^2}+\frac{1}{r}\frac{dR}{dr}-\frac{2}{r}\frac{dT}{dz}\right\}=\sigma\left(\frac{d^2}{dz^2}+\frac{1}{r}\frac{d}{dr}\right)\left\{P+Q+R\right\},$$

$$(1+\sigma)\left\{r\frac{dQ}{dr}+Q-P\right\}=\sigma r\frac{d}{dr}\left\{P+Q+R\right\},$$

$$\frac{dS}{dr}-\frac{S}{r}=\frac{dU}{dz}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

These are the necessary and sufficient conditions for the elastic relations.

If we proceed to find the statical stress which is consistent with Hooke's Law, we combine the two sets of conditions. We may write

$$P = -\frac{1}{r}\frac{d\mu}{dr} + \frac{1}{r^2}\mu + P'$$

whence we must have from (13)

$$Q = -\frac{d^2\mu}{dr^2} + \frac{1}{r}\frac{d\mu}{dr} - \frac{1}{r^2}\mu - \frac{d^2\mu}{dz^2} + r\frac{dP'}{dr} + P'.$$

Using now the second set of conditions (14), we find

$$r^2\frac{d^2P'}{dr^2} + 3r\frac{dP'}{dr} = \left(r\frac{d}{dr} + 1 + \sigma\right)\left\{\frac{d^2\mu}{dr^2} - \frac{1}{r}\frac{d\mu}{dr} + \frac{d^2\mu}{dz^2}\right\}$$

and

$$\frac{d^2}{dz^2}\left\{r\frac{dP'}{dr} + (1-\sigma)P\right\} = \left(\frac{d^2}{dz^2} - \frac{1-\sigma^2}{r^2}\right)\left\{\frac{d^2\mu}{dr^2} - \frac{1}{r}\frac{d\mu}{dr} + \frac{d^2\mu}{dz^2}\right\},$$

giving two equations connecting P' and $\frac{d^2\mu}{dr^2} - \frac{1}{r}\frac{d\mu}{dr} + \frac{d^2\mu}{dz^2}$.

If each of these quantities is null, the stress is independent of the elastic constants of the material.

The general case where the stress is independent of the elastic constants seems worthy of consideration. The distribution of the stress would in this case be geometrical, but the strain would depend on the elastic constants.

On the modes of rupture of an open hemispherical concrete shell under axial pressure.

BY

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IX. On the modes of rupture of an open hemispherical concrete shell under axial pressure.

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(*Communicated by Mr. R. F. Gwyther, M.A.*)

(*Received and read 23rd January, 1912.*)

This paper is written to describe a few experiments undertaken tentatively in the hopes of obtaining some definite description of the circumstances and mode of rupture as the load is increased, and it is the author's intention to continue the investigation. The specimens experimented on were made of concrete in the proportion of 1 : 1½ : 2 of cement, sand and stone, the aggregate being ¼-inch granite chippings, and were carefully prepared in wood moulds. After remaining in the moulds for seven days, they were removed and allowed to set under water for a month, and were finally tested four days after being removed from the water. The specimens were then subjected in the ordinary way to compression in a horizontal testing machine with the results to be described. It would perhaps have been preferable to have used a vertical machine for the purpose, since when the load was not removed sufficiently quickly on causing rupture, the specimens were injured by the falling parts.

The author wishes to express his acknowledgments to Mr. J. H. Reynolds, the Principal of the Manchester Municipal School of Technology, to Mr. Popplewell, and to Mr. A. Herring-Shaw for permission to use the laboratories and apparatus.

Six specimens were made and tested, of three sizes with two specimens of each, and at a later period three more specimens were tested. One specimen of each size was reinforced at the base (1a, 2a, 3a in the Table, p. 7). The other specimens of each size were not reinforced (marked 1, 2, and 3 in the Table, p. 7).

The several specimens and the results of the tests are described below, but it appears best to state first the general modes of rupture as the load increases.

First. At some load a longitudinal crack develops which extends gradually in the meridian plane. There are several such cracks fairly regularly distributed, but no doubt decided in position at first by some accidental weakness. When once started they doubtless affect the condition of the specimen.

In the specimens not reinforced at the base, the cracks started at the base; in those which were reinforced at the base they commenced at the top.

Secondly. When the load was increased, and (except in the case of the two smaller specimens) before the longitudinal cracks had extended through the material, rupture took place quite suddenly by a fracture roughly along a parallel of latitude. This fracture was approximately conical and nearly normal to the spherical surfaces, although in all cases the vertex of the cone appeared to be in the axis slightly below the centre of the sphere.

It must be understood that the fracture was irregular, and that the description is of its general character.

Description of the specimens and their rupture.

The specimens were all of the same description of concrete, and the difference between the radii of the bounding spheres was in each case one inch, so that if d is the diameter of the internal sphere in inches, the area of any section parallel to the base is $\pi(d+1)$ square inches. The vertical breaking stress is found in pounds per square inch by dividing the breaking load by the area of the section.

The top was plane and parallel to the base.

The specimens I. and Ia. had the dimensions:—

internal diameter, 12·5 inches,
external diameter, 14·5 inches,
height, 5·75 inches,

and Ia. was reinforced at the base with two rings of wire one-sixteenth inch diameter, a quarter of an inch from the bottom.

I. The longitudinal cracks first appeared under a load of 3·35 tons, and spread up the dome as the load increased, reaching about three-quarters or four-fifths the height when the specimen broke latitudinally under the load of 4·42 tons, or a vertical breaking stress of 233·6 lbs. per sq. inch. Care was taken in removing the specimen from the machine, the lower portion coming away in pieces, while the top remained intact.

Ia. The longitudinal cracks first appeared under the load of 4·01 tons, and in this case started from the top, spreading gradually downwards. When the cracks had reached about two inches from the base the specimen broke latitudinally, as in the previous case, under the load of 11·11 tons, or a vertical breaking stress of 586·7 lbs. per sq. inch. Part of the top collapsed, but the bottom remained whole.

Specimens 2. and 2a. had the following dimensions:—

internal diameter, 9 inches.
external diameter, 11 inches.
height, 4·25 inches,

and 2a. was reinforced at the base with two rings of one-sixteenth inch diameter wire.

2. Longitudinal cracks first appeared under a load of 2·6 tons, commencing at the base and spreading upwards, but did not reach the top, being higher on the outside than the inside. The latitudinal rupture took place under the load of 4·6 tons and was irregular, the vertical breaking stress in this case being 328 lbs. per sq. inch. On removing the specimen from the machine the lower portion was found to be in sections, but the top held together.

2a. Longitudinal cracks appeared under a load of 7·25 tons and spread from the top downwards as the load increased. At the load of 10·68 tons the specimen broke latitudinally, the top collapsed, and, falling inwards, broke a part of the base to the level of the reinforcement, the longitudinal cracks not having reached the reinforcement. The breaking stress was in this case 761·4 lbs. per sq. inch. The load was not removed sufficiently quickly, and the injury to the specimen was partly due to testing it in a horizontal machine. The latitudinal crack was irregular but roughly normal to the surface.

The dimensions of specimens 3. and 3a. were as follows:—

internal diameter, 6 inches,
external diameter, 8 inches,
height, 2·75 inches,

and 3a. was reinforced at the base.

3. In this case the longitudinal cracks appeared under a load of 3·36 tons. They spread upwards and the specimen parted in sections, three to four inches in width at the base, under a load of 4·72 tons or a vertical breaking stress of 481 lbs. per sq. inch. There was no latitudinal crack.

3a. This specimen was reinforced at the base, and longitudinal cracks first appeared under a load of 5·2 tons, and spread from the top downwards as the load increased. At the load of 12·32 tons, giving a breaking stress of 1254·9 lbs. per sq. inch, the specimen collapsed, shearing off at the base, just above the level of the reinforcement, leaving only the inner half of the thickness of the base with the reinforcement standing. There was no latitudinal crack.

Having experimented on domes of three different sizes, first without reinforcement, then with reinforcement at the base to prevent spreading, I next decided to test further specimens of the same size as before, but reinforced with wire rings both at the top and bottom to prevent any spreading movement starting either at the top or bottom with the results and the modes of rupture described below.

General mode of rupture of specimens reinforced top and bottom.

Firstly. At some load cracks developed in meridian planes round the middle of the dome, spreading up and down as the load increased.

Secondly. When the load was increased and before the cracks had extended to the top or bottom, rupture took place quite suddenly by an irregular fracture roughly along a parallel of latitude, and, as in the case of the former experiments, was approximately conical, and nearly normal to the spherical surfaces, although in each case the vertex of the cone appeared to be in the axis and slightly below the centre of the sphere. As the rupture took place there was an extension of the meridional cracks.

1b. The longitudinal cracks first appeared under the load of 6·03 tons at intervals averaging roughly two inches round the middle of the dome, and spread gradually up and down. When these cracks had almost reached the top and bottom, the specimen broke latitudinally under the load of 20·35 tons, giving a breaking stress of 1074·8 lbs. per sq. inch, the average height being about three inches. The specimen was removed from the machine in two pieces.

2b. Longitudinal cracks appeared round the middle as in 1b., in this case under the load of 7·12 tons, spreading gradually as the load increased. At the load of 20·63 tons, a breaking stress of 1470·7 lbs. per sq. inch, the specimen broke latitudinally, the longitudinal cracks not having reached the reinforcements at the top or base. Again the latitudinal crack was irregular and was at an approximate mean height of 2¾ inches from the bottom.

3b. Longitudinal cracks first appeared in meridian planes under a load of 8·5 tons and spread as in the other cases, but ultimately reached both reinforcements, the whole breaking and falling to pieces under a load of 21·2 tons, having sheared through at the level of the reinforcements. The breaking stress in this case was 2159·4 lbs. per sq. inch. The broken pieces appeared to indicate that the specimen had fractured roughly along a parallel of latitude just as the longitudinal cracks reached the reinforcements, the whole collapsing instantaneously.

Table showing Vertical Breaking Loads in Tons and Breaking Stresses in Pounds per square inch.

PLAIN.

No. of Specimen.	Load.	Stress.
1.	4·42	233·6
2.	4·6	328·0
3.	4·72	481·0

REINFORCED AT BASE.

1a.	11·11	586·7
2a.	10·68	761·4
3a.	12·32	1254·9

REINFORCED BOTH AT BASE AND CROWN.

1b.	20·35	1074·8
2b.	20·63	1470·7
3b.	21·2	2159·4

Summary of the results.

(1) The longitudinal cracks in meridional planes only develop under considerable loads and extend slowly. As concrete is understood not to be able

to resist tension, the conclusion must be that the "ring tension" is comparatively small. The theory of the "Angle of Rupture" does not apply to cases of externally applied load, and it probably is not applicable even to a concrete dome under its own weight.

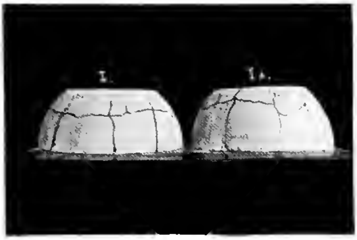
(2) Whatever the size of the example on which the experiments have been carried on, the fracture has occurred under the same conditions (except in the case of the smallest specimens) for practically the same load, and not for the same stress. It may be concluded that the cause of the fracture is not 'shear' under which concrete is supposed to be apt to break, but it breaks in consequence of an excessive 'bending moment,' or otherwise stated, that the resultant stress on some section fails to act within the 'middle third.'

(3) The section in which the fracture takes place appears to be approximately on a cone of which the centre of the sphere is the vertex. The mean height of the fracture is about ·66 of the height of the shell.

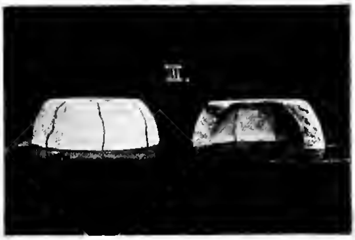
(4) The angle of the cone does not vary very greatly with the reinforcement given to the concrete.

(5) The load which the specimen will bear without rupture is greatly increased by reinforcement at the lower rim, and is again greatly increased by reinforcement at both top and bottom.

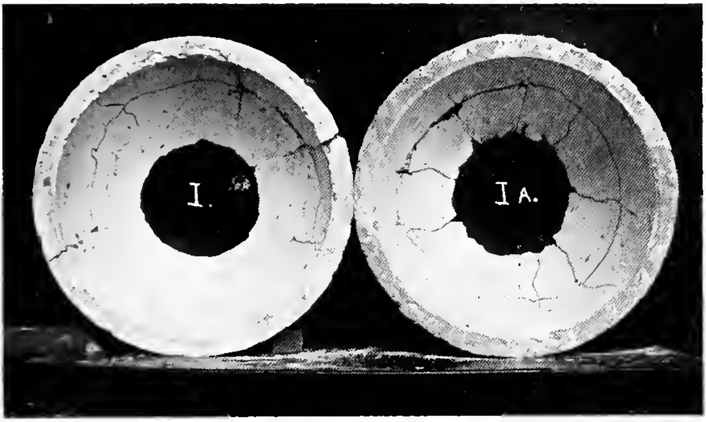
It is desirable that a greater number of experiments should be made, as it is probable that a very slight displacement of the reinforcing wires may make a very considerable difference in the load which the specimen can bear without fracture.



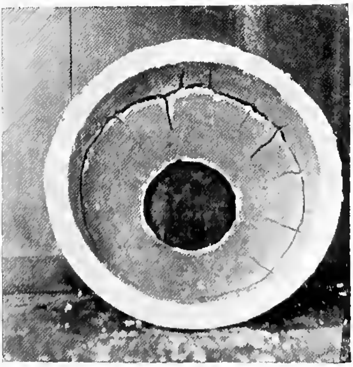
Specimen 1 and 1a. (Elevation.)



Specimen 3. (Elevation and Section.)



Specimen 1 and 1a. (Interior.)



Specimen 2b. (Interior.)

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